Radiative Transfer through an Isotropically Scattering Finite Medium: Approximate Solution

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The influence of substrate reflectance on the radiative intensity and flux distribution and on the apparent reflectance and transmittance of an isothermal, isotropically scattering medium has been examined by using an exponential kernel approximation to obtain a simple, closed-form, approximate solution for the governing radiative transport equation. The approximate method provides one simple algebraic expression from which the bidirectional, directional hemispherical, hemispherical directional, and hemispherical reflectances can be evaluated with reasonable accuracy (better than 20%) over a wide range of optical parameters. The intensity and flux distribution also have been investigated for both conservative and nonconservative cases.

Introduction

RADIATIVE transfer through a semitransparent medium is affected by the absorption and scattering characteristics of the medium, the angular distribution of the incident radiation, and the reflection characteristics of the bounding surfaces. Most of the published results deal with either a medium that has two transparent boundaries or one that has a transparent upper boundary and a black substrate. Crosbie[1] used the technique developed by Chandrasekhar[2] to examine the reflection characteristics of an isotropically scattering medium with a black substrate. This technique provides exact and simple expressions for the reflectance and the transmittance of such a medium (boundary values), but does not provide a simple solution for radiative transfer through the medium. Lii and Özisik[3] determined, through the means of an approximate solution, the hemispherical reflectance and transmittance for the case of a black substrate and a perfectly reflecting substrate. Their approximate solution provides physically meaningless results (negative reflectances) when the scattering albedo is smaller than 0.2. The effect of refractive index and substrate reflectance was examined through an exact numerical solution by Roux and Smith.[4] The reviews and discussions of related references can be found in the papers by Crosbie[1] and by Lii and Özisik.[3]

An absorbing, isotropically scattering, conservative (and nonconservative), plane parallel slab of an optical thickness of τ_0 is the subject of the present study. It is assumed that collimated and diffuse radiation is incident at the upper transparent boundary $\tau = 0.0$, and that the boundary at $\tau = \tau_0$ is a diffuse reflector. The directional and hemispherical reflectances, transmittance, intensity, and flux distribution of this case are examined through an approximate solution. Simple, algebraic, closed-form expressions are developed to describe the behavior of these quantities as a function of direction, scattering albedo, substrate reflectance, optical thickness, and optical depth. Comparison, when possible, with exact solutions indicates reasonable agreement. In combined radiation-conduction and convection problems, in which nongray behavior is important, the exact numerical treatment[4] of the radiation components can significantly complicate, and in some cases make the solution economically unfeasible. In these cases, the derived approximate solution can provide tremendous savings in calculation time without sacrificing greatly on accuracy.

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Formulation

The test case consists of a finite planar medium of unity refractive index which absorbs and scatters radiation isotropically, and is bounded by a diffusely reflecting substrate. It is exposed to collimated and diffuse incident radiation on its upper smooth and transparent boundary, as shown in Fig. 1. The emission of thermal radiation from both the substrate and the medium at the same wavelengths as the incident radiation is considered to be negligible. The results and formulation that are presented below are applicable on a monochromatic basis. The medium is characterized by a scattering albedo ω, which is the ratio of scatter σ to extinction coefficient β, optical thickness $\tau_0 = \int_0^L \beta \mathrm{d}x$, and optical depth $\tau = \int_0^x \beta \mathrm{d}x$. The intensity distribution within the medium is given by[5]

$$I^+(\tau,\mu,\phi) = I^+(0,\mu,\phi)\exp(-\tau/\mu) + \int_0^\tau S(t,\tau_0)\exp[-(\tau-t)/\mu]\mathrm{d}t/\mu \tag{1}$$

and

$$I^-(\tau,\mu,\phi) = I^-(\tau_0,\mu,\phi)\exp[-(\tau_0-\tau)/\mu] + \int_\tau^{\tau_0} S(t,\tau_0)\exp[-(t-\tau)/\mu]\mathrm{d}t/\mu \tag{2}$$

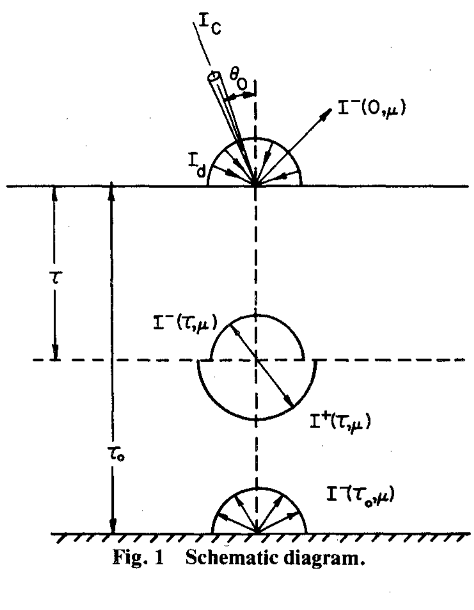

Fig. 1 Schematic diagram.

in which

$$S(\tau,\tau_0) = (\omega/4\pi)[\int_0^{2\pi}\int_0^1 I^+(\tau,\mu,\phi)\mathrm{d}\mu\mathrm{d}\phi + \int_0^{2\pi}\int_0^1 I^-(\tau,\mu,\phi)\mathrm{d}\mu\mathrm{d}\phi] \quad (3)$$

The boundary conditions are given by

$$I^+(0,\mu,\phi) = I_d + F_c\delta(\mu-\mu_0)\delta(\phi-\phi_0) \quad (4)$$

and

$$I^-(\tau_0,\mu,\phi) = R(\tau_0)/\pi \quad (5)$$

The radiosity $R(\tau_0)$ for the diffuse substrate is given by

$$R(\tau_0) = \rho_b\int_0^{2\pi}\int_0^1 I^+(\tau_0,\mu,\phi)\mu\mathrm{d}\mu\mathrm{d}\phi \quad (6)$$

The terms $I^+(\tau,\mu,\phi)$ and $I^-(\tau,\mu,\phi)$ denote the intensities in the plus and minus directions, respectively; $S(\tau,\tau_0)$ is the isotropic source function, I_d and F_c represent the diffuse and collimated radiation incident on the interface, and ρ_b is the hemispherical reflectance of the diffuse substrate. Because of the linearity of the governing equations and the boundary conditions, the influence of diffuse and incident radiation can be treated independently and then superimposed to obtain the solution of the general combined problem. By using this feature and making appropriate substitutions in Eq. (3), the source function can be expressed as

$$S(\tau,\tau_0) = S_c(\tau,\tau_0) + S_d(\tau,\tau_0) \quad (7)$$

$$S_c(\tau,\tau_0) = (F_c\mu_0/\pi)\{[\omega p(\tau,\tau_0,\mu_0)/\mu_0] + [R_c(\tau_0)/\mu_0F_c]\omega F(\tau_0-\tau,\tau_0)\} \quad (8)$$

and

$$S_d(\tau,\tau_0) = I_d\{\omega F(\tau,\tau_0) + [R_d(\tau_0)/\pi I_d]\omega F(\tau_0-\tau,\tau_0)\} \quad (9)$$

in which the subscripts c and d on S and R refer to the association of those parameters in accordance with either collimated or diffuse incident radiation. The functions $p(\tau,\tau_0,\mu_0)$ and $F(\tau,\tau_0)$ are the dimensionless source functions associated with collimated and diffuse incident radiation, respectively, and are given by

$$p(\tau,\tau_0,\mu_0) = [\exp(-\tau/\mu_0)/4 + (\omega/2)\int_0^{\tau_0} p(t,\tau_0,\mu_0)E_1(|\tau-t|)\mathrm{d}t \quad (10)$$

and

$$F(\tau,\tau_0) = [E_2(\tau)/2] + (\omega/2)\int_0^{\tau} F(t,\tau_0)E_1(|\tau-t|)\mathrm{d}t \quad (11)$$

The functions $E_1(t)$ and $E_2(t)$ are the exponential integrals defined by

$$E_n(t) = \int_0^1 \exp(-\tau/\mu)\mu^{n-2}\mathrm{d}\mu \quad (12)$$

The solutions to the preceding equations and their use in Eqs. (1) and (2) provide the intensity distribution within the medium. The directional and hemispherical reflectance and transmittance can be evaluated from the appropriate definitions.[6]

The numerical solution of the previously outlined system of equations is lengthy and complicated. When applied to nongray radiative transfer problems, these equations need to be solved for each wavelength and location, and then integrated over the total wavelength spectrum in order to evaluate total properties or energy transfer through the medium. The time required for exact numerical solution becomes prohibitive. In the following section, an approximate but closed-form solution is presented for the previous system of equations which agrees closely with the exact solution and greatly speeds the required calculations. The proposed method of solution, exponential kernel approximation, has been used on several radiative transfer problems, and has been shown to provide a reasonable approximation.[5] This technique has not been employed previously to examine the present problem. The results that will be generated in this manuscript are not available in the literature, and as stated earlier, they are extremely useful for evaluating radiant energy transfer through a finite isotropically scattering medium.

Approximate Solution

The use of an exponential kernel approximation for the exponential integral given by $E_1(t) = q\exp(-qt)$ provides a closed-form approximate solution for the two dimensionless source functions that appear in Eqs. (10) and (11). The solutions are given by[7]

$$p(\tau,\tau_0,\mu_0) = C_1\exp(-b\tau) + C_2\exp(b\tau) + C_3\exp(-\tau/\mu_0) \quad (13)$$

and

$$F(\tau,\tau_0) = C_4\exp(-b\tau) + C_5\exp(b\tau) \quad (14)$$

in which the constants are

$$b^2 = q^2(1-\omega) \quad (15)$$

$$C_1 = C_2(b-q)/(b+q) - C_3[\mu_0(b-q)]/(1-\mu_0q) \quad (16)$$

$$C_2 = C_3\mu_0q^2\omega(\{(b-q)(1+\mu_0q)\exp(-2b\tau_0) + (b+q)(1-\mu_0q)\exp[-\tau_0(b+1/\mu_0)]\} \div \{[(b+q)^2-(b-q)^2\exp(-2b\tau_0)](1-\mu_0{}^2q^2)\}) \quad (17)$$

$$C_3 = (1-\mu_0{}^2q^2)/[4(1-\mu_0{}^2b^2)] \quad (18)$$

$$C_4 = C_5(b+q)\exp(2b\tau_0)/(b-q) \quad (19)$$

and

$$C_5 = q(b-q)\exp(-2b\tau_0)/[(b+q)^2-(b-q)^2\exp(-2b\tau_0)] \quad (20)$$

For the conservative case, $\omega = 1$, the solutions for the two integral equations take the following simple forms:

$$F(\tau,\tau_0) = [1+q(\tau_0-\tau)]/(2+q\tau_0) \quad (21)$$

and

$$p(\tau,\tau_0,\mu_0) = 0.25(1+\mu_0q)[\mu_0q + (1-\mu_0q)\exp(-\tau/\mu_0)] - [0.25\mu_0q(q\tau+1)/(q\tau_0+2)][(1+\mu_0q) + (1-\mu_0q)\exp(-\tau_0/\mu_0)] \quad (22)$$

It has been shown[7] that this approximate solution agrees closely with the exact solutions.[8,9] It is easy to see that when the coefficient $q = 2$ is used in the approximation of the exponential integral, the functions become related to each other

$$F(\tau,\tau_0) = 2p(\tau,\tau_0,0.5) \quad (23)$$

This fact eliminates the need for solving the case with diffuse incident radiation because that result can be obtained directly from the case of collimated incident radiation when the incidence angle is 60 deg, or when $\mu_0 = 0.5$.

By using the approximate solution, the bidirectional reflectance $\rho(\mu,\mu_0)$, which is defined by[1]

$$\rho(\mu,\mu_0) = \pi I^-(0,\mu,\phi)/\mu_0 F_c \tag{24}$$

can be expressed as

$$\begin{aligned}\rho(\mu,\mu_0) = {} & \bar{R}(\tau_0,\mu_0)\{\exp(-\tau_0/\mu) + \omega\{C_4[\exp(-\tau_0/\mu) \\ & - \exp(-b\tau_0)]/(\mu b - 1) + C_5[\exp(b\tau_0) - \exp(-\tau_0/\mu)] \\ & \div (\mu b + 1)\} + (\omega/\mu_0)(C_1\{1 - \exp[-(b + 1/\mu)\tau_0]\} \\ & \times (\mu b + 1) + C_2\{\exp[(b - 1/\mu)\tau_0] - 1\}/(\mu b - 1) \\ & + \mu_0 C_3\{1 - \exp[-(1/\mu_0 + 1/\mu)\tau_0]\}/(\mu + \mu_0))\end{aligned} \tag{25}$$

The dimensionless radiosity $\bar{R}(\tau_0,\mu_0)$ is governed by Eq. (6) for the case of collimated incident radiation, and is given by

$$\bar{R}(\tau_0,\mu_0) = R_c(\tau_0)/(\mu_0 F_c) = A/B \tag{26}$$

in which

$$\begin{aligned}A = {} & \rho_b\{\exp(-\tau_0/\mu_0) + (2\omega/\mu_0)C_1[\exp(-q\tau_0) \\ & - \exp(-b\tau_0)]/(b - q) + C_2[\exp(b\tau_0) - \exp(-q\tau_0)] \\ & \div (b + q) + \mu_0 C_3[\exp(-q\tau_0) - \exp(-\tau_0/\mu_0)]/(1 - \mu_0 q)\}\end{aligned} \tag{27}$$

and

$$\begin{aligned}B = {} & 1 - 2\omega\rho_b(C_4\{1 - \exp[-(b + q)\tau_0]\}/(b + q) \\ & + C_5\{\exp[(b - q)\tau_0] - 1\}/(b - q))\end{aligned} \tag{28}$$

The radiative flux distribution within the medium, which results from collimated radiation that is incident in the direction of μ_0, is given by

$$F^+(\tau,\tau_0,\mu_0) = \int_0^{2\pi}\int_0^1 I^+(\tau,\mu,\phi)\,\mu\,d\mu\,d\phi \tag{29}$$

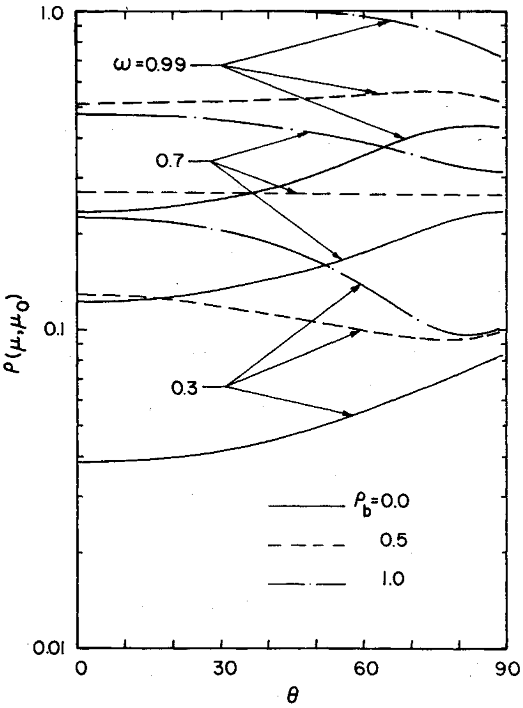

Fig. 3 Bidirectional reflectance for $\tau_0 = 1$ and $\mu_0 = 1$.

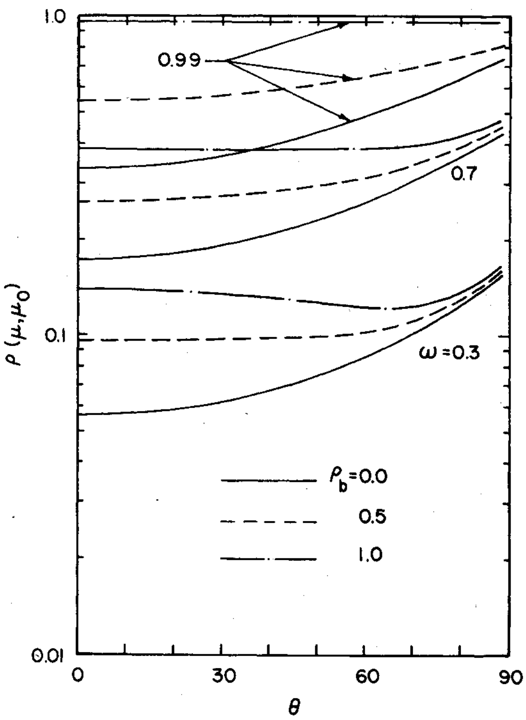

Fig. 4 Bidirectional reflectance for $\tau_0 = 1$ and $\mu_0 = 0.5$.

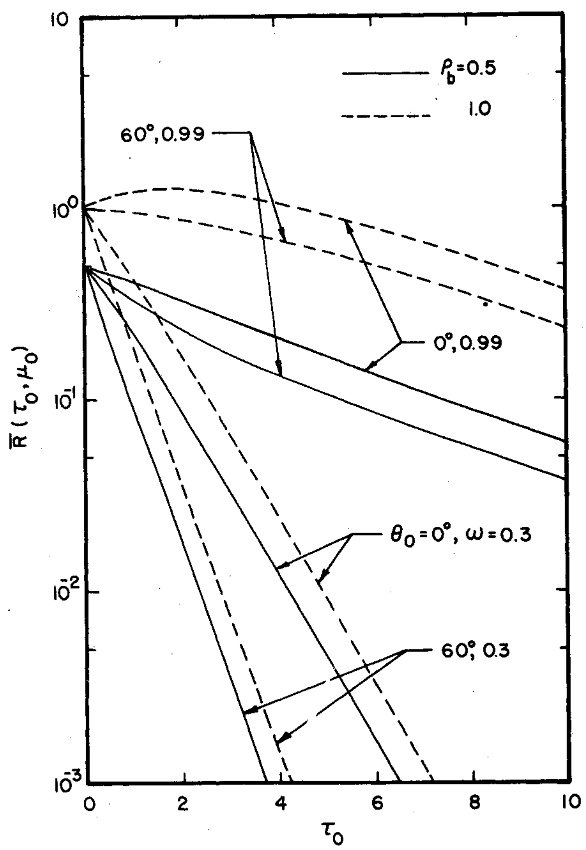

Fig. 2 Dimensionless radiosity for the substrate.

and

$$F^-(\tau,\tau_0,\mu_0) = \int_0^{2\pi}\int_0^1 I^-(\tau,\mu,\phi)\,\mu d\mu d\phi \qquad (30)$$

The approximate solution provides the following expressions for the dimensionless fluxes

$$\begin{aligned} D^+(\tau,\tau_0,\mu_0) &= [F^+(\tau,\tau_0,\mu_0)/\mu_0 F_c] = \exp(-\tau/\mu_0) \\ &+ 2\omega\bar{R}(\tau_0,\mu_0)\{C_4[\exp(-q\tau) - \exp(-b\tau)]/(b-q) \\ &+ C_5[\exp(b\tau) - \exp(-q\tau)]/(b+q)\} + (2\omega/\mu_0) \\ &\times \{C_1[\exp(-q\tau) - \exp(-b\tau)]/(b-q) + C_2[\exp(b\tau) \\ &- \exp(-q\tau)]/(b+q) + C_3\mu_0[\exp(-q\tau) \\ &- \exp(-\tau/\mu_0)]/(1-\mu_0 q)\} \end{aligned} \qquad (31)$$

and

$$\begin{aligned} D^-(\tau,\tau_0,\mu_0) &= [F^-(\tau,\tau_0,\mu_0)/\mu_0 F_c] = 2\bar{R}(\tau_0,\mu_0) \\ &\times (E_3(\tau_0-\tau) + \omega\{C_4[\exp(-b\tau) - \exp[-(b+q)\tau_0] \\ &\times \exp(q\tau)\}/(b+q) + C_5\{\exp[(b-q)\tau_0]\exp(q\tau) \\ &- \exp(b\tau)\}/(b-q)) + (2\omega/\mu_0)(C_1\{\exp(-b\tau) \\ &- \exp[-(b+q)\tau_0]\exp(q\tau)\}/(b+q) + C_2\{\exp[(b-q)\tau_0] \\ &\times \exp(q\tau) - \exp(-b\tau)\}/(b-q) + \mu_0 C_3\{\exp(-\tau/\mu_0) \\ &- \exp[-(q+1/\mu_0)\tau_0]\exp(q\tau)\}/(1+\mu_0 q)) \end{aligned} \qquad (32)$$

The directional hemispherical reflectance[1] is given by

$$\rho(\mu_0) = D^-(0,\tau_0,\mu_0) \qquad (33)$$

This property also can be evaluated directly from the expression for the bidirectional reflectance [Eq. (25)] as

$$\rho(\mu_0) = \rho(0.5,\mu_0) \qquad (34)$$

The directional hemispherical transmittance is given by

$$T(\tau,\tau_0,\mu_0) = D^+(\tau,\tau_0,\mu_0) - D^-(\tau,\tau_0,\mu_0) \qquad (35)$$

As previously stated, when this approximate method is applied to the case of diffuse incident radiation $[I^+(0,\mu,\phi) = I_d]$, the results can be deduced directly from those presented previously for the collimated incident radiation case. The subscript d is used to identify the properties associated with the diffuse case. The hemispherical directional reflectance[1] is given by

$$\rho_d(\mu) = I^-(0,\mu,\phi)/I_d = \rho(\mu,0.5) \qquad (36)$$

The dimensionless radiosity appearing in Eq. (25) should be replaced by its equivalent for the diffuse case, which is given by

$$\bar{R}_d(\tau_0) = R_d(\tau_0)/\pi I_d = \bar{R}(\tau_0,0.5) \qquad (37)$$

The dimensionless flux distribution for this case is given by

$$F_d^{\,+}(\tau,\tau_0)/\pi I_d = D^+(\tau,\tau_0,0.5) \qquad (38)$$

and

$$F_d^{\,-}(\tau,\tau_0)/\pi I_d = D^-(\tau,\tau_0,0.5) \qquad (39)$$

The hemispherical reflectance[1] is given by

$$\rho_d = \rho(0.5,0.5) \qquad (40)$$

and the hemispherical transmittance by

$$T_d(\tau,\tau_0) = T(\tau,\tau_0,0.5) \qquad (41)$$

Results and Discussion

Results are presented for the collimated incident radiation case, and the special case of $\mu_0 = 0.5$ is included to facilitate the evaluation of the diffuse incident radiation case. The dimensionless radiosity is presented in Fig. 2, and as expected it increases as the substrate reflectance and the scattering albedo increase. For most of the cases, it decreases as the optical thickness and incident angle increase. An exception occurs when the substrate reflectance is large, $\rho_b = 1$, the scattering albedo is large, $\omega \approx 1$, and the incident angle is small $\mu_0 = 1$. These conditions cause the radiosity to increase slightly with optical thickness to a maximum value larger than unity at $\tau_0 \approx 2$. It then decreases as the optical thickness continues to increase. This behavior is in agreement with what is predicted from exact solution.[9] The bidirectional reflectance increases as the substrate reflectance and scattering albedo increase, as can be seen from Figs. 3 and 4. The influence of the substrate reflectance is more pronounced at higher scattering albedo and at smaller incident angles. This is because these conditions produce a smaller effective optical thickness, which is proportional to $\tau_{0\text{eff}} \propto (1-\omega)\tau_0/\mu_0\mu$. When this parameter increases, the influence of the substrate reflectance diminishes, as demonstrated in Fig. 5. A comparison with the exact solution[1] for the case of a black substrate is presented in Fig. 5. The agreement is better than 15%. The directional hemispherical reflectance is presented in Fig. 6 as a function of optical thickness, and it is similar in characteristics to the bidirectional reflectance. As the effective optical depth increases, the influence of the substrate diminishes and the directional hemispherical reflectance approaches the asymptotic value given by $(1-\sqrt{1-\omega})/(1+2\mu_0\sqrt{1-\omega})$. The hemispherical reflectance, which can be deduced either from Fig. 4, $\rho_d = \rho(0.5, 0.5)$, or Fig. 6, $\rho_d = \rho(0.5)$, is presented in Fig. 7, and it is always higher than the directional hemispherical reflectance, which was presented in Fig. 6. A comparison with the available exact solution[3] is presented in Fig. 7. A better agreement exists for large scattering albedo, when the substrate reflectance is large. When the scattering albedo is small, the agreement improves as the substrate reflectance decreases. It should be noted that in the region of $\omega < 0.2$ where this approximation produces the largest error (about 20%), the P_1 approximation[3] produces negative reflectances.

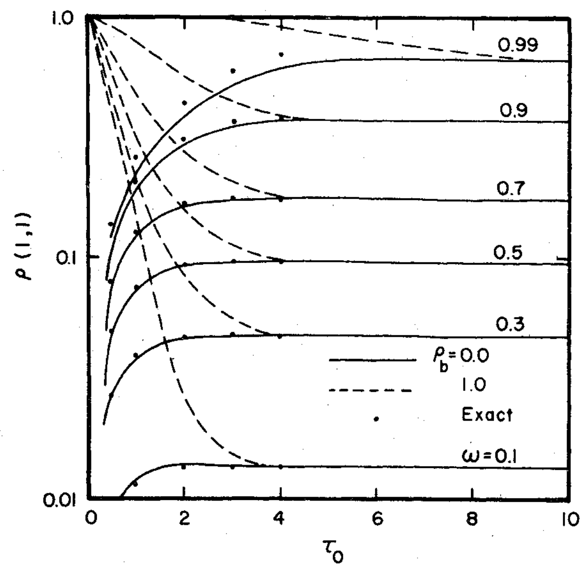

Fig. 5 Influence of optical thickness on bidirectional reflectance.

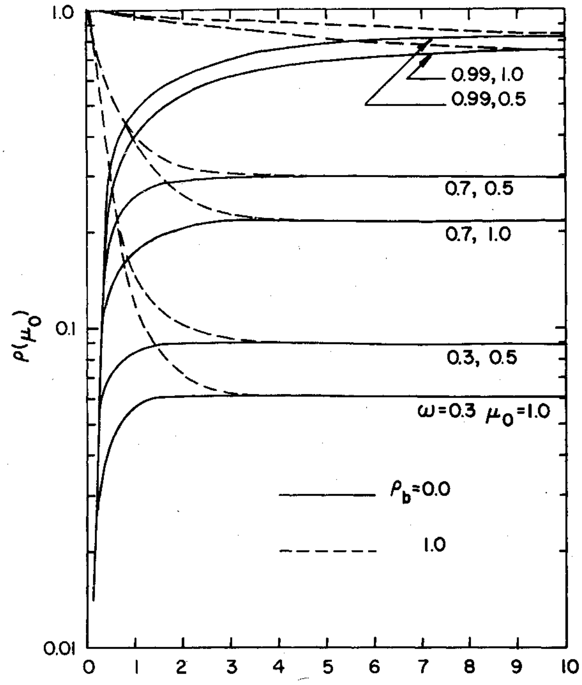

Fig. 6 Influence of optical thickness on directional hemispherical reflectance.

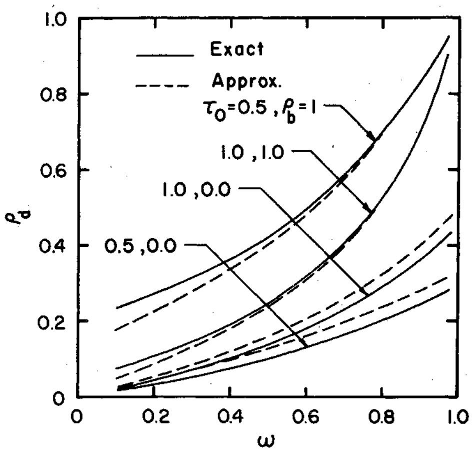

Fig. 7 Comparison between exact and approximate hemispherical reflectance.

The fluxes in both the positive and negative directions increase as the scattering albedo and the substrate reflectance increase as shown in Figs. 8 and 9. The positive flux decreases almost exponentially with optical depth, except in cases in which the scattering albedo and substrate reflectance are large, and the incident angle is small. Its magnitude decreases as the incident angle increases. The negative flux, which always is smaller than the positive flux, can either increase or decrease (not exponentially) with optical depth depending on the ratio of the radiosity to the scattered induced flux. The transmittance, which is shown in Fig. 10, decreases as the substrate reflectance increases. When the substrate reflectance is zero, it can be approximated by one exponentially decaying term. That approximation is not appropriate when the substrate is other than black.

Conclusion

It has been shown that substrate reflectance does influence the radiative transfer and properties of an isothermal, isotropically scattering medium. The bidirectional, hemispherical-directional, directional-hemispherical, and

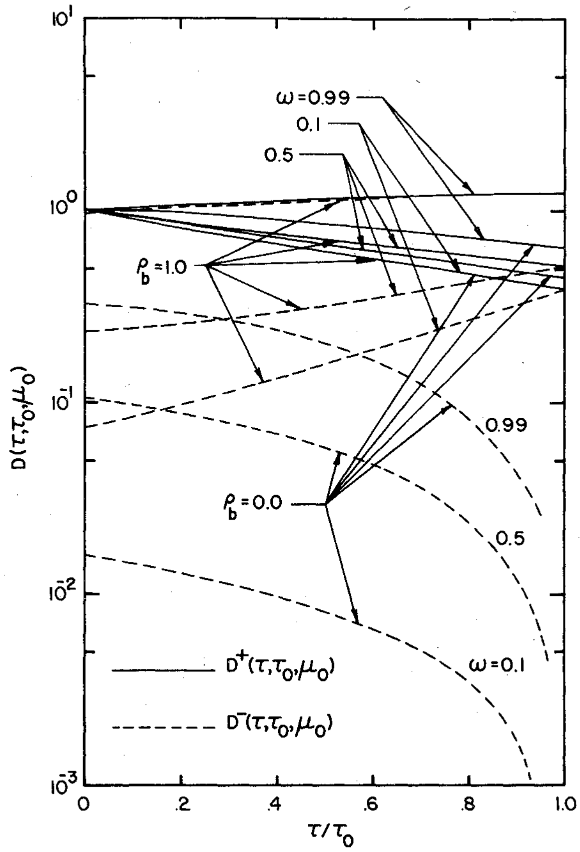

Fig. 8 Flux distribution for $\tau_0 = 1$ and $\mu_0 = 1$.

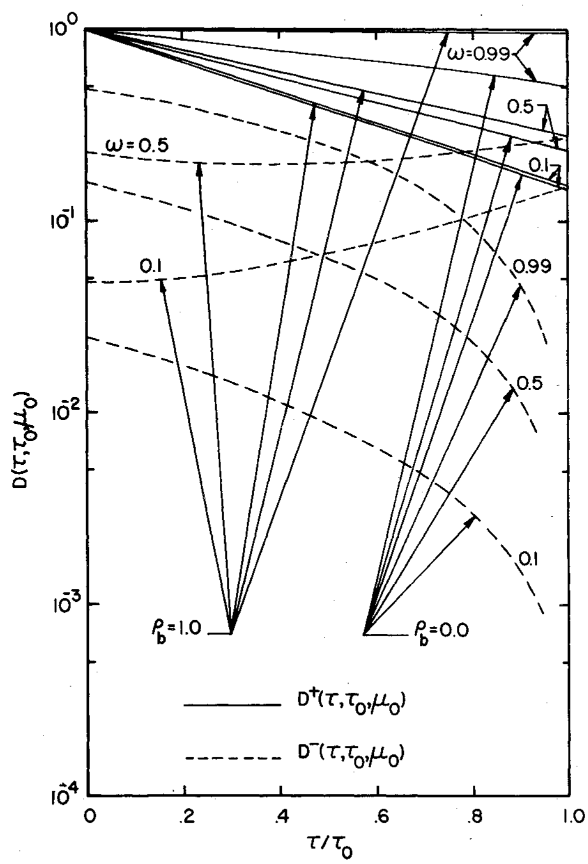

Fig. 9 Flux distribution for $\tau_0 = 1$ and $\mu_0 = 0.5$

hemispherical reflectance can be approximated by using the same simple algebraic expression, which is given in Eq. (25), $\rho(\mu_0) = \rho(0.5, \mu_0)$, $\rho_d(\mu) = \rho(\mu, 0.5)$, and $\rho_d = \rho(0.5, 0.5)$.

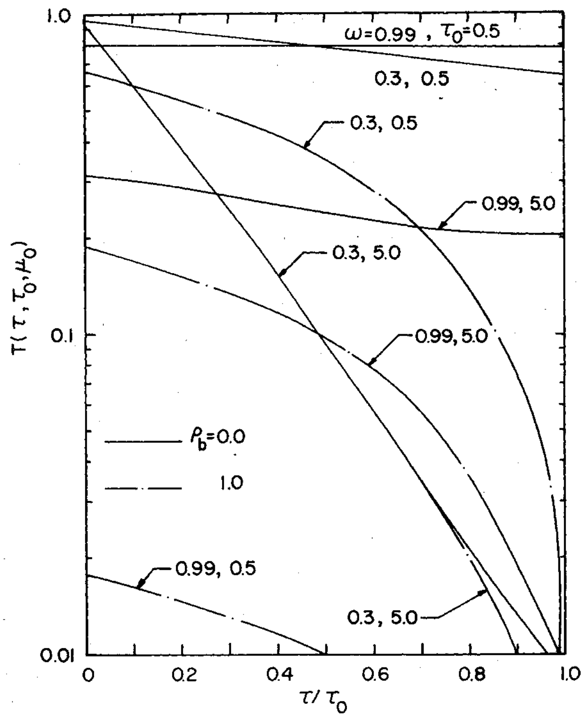

Fig. 10 Directional hemispherical transmittance, $\mu_0 = 1$.

Similarly, one algebraic expression, Eq. (35), could be used to evaluate both the hemispherical and the directional hemispherical transmittances, $T_d(\tau,\tau_0) = T(\tau,\tau_0, 0.5)$. The approximate method produces errors of less than 20% over the entire range of optical parameters, and it provides an economical means of examining in detail the radiative transfer through such a medium.

Acknowledgment

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